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Unit-1

Linear System of Equations

Real and complex matrices and linear system of Equation

Matrix Definition:-

A System of mn numbers (real and complex) arranged in the form of an ordered set of m rows, each row consisting of an ordered set of n numbers between [ ] or ( ) or || || is called a matrix of order (or) type m×n

Each of mn numbers constituting the m×n matrix is called an element of the matrix. Thus we write a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{m\times n}$$

where
$1 \le j \le m$
$1 \le j \le n$

In relation to a matrix, we call the numbers as a scalars.

Type of Matrices

Definition:-

1. If $A = [a_{ij}]_{m\times n}$ and $m=n$, then A is called a square matrix. A square matrix A of order

$n \times n$ is something called as a n-rowed matrix A (or) simply a square matrix of order n.

Eg:- $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2nd order matrix

2. A matrix which is not a square matrix is called a rectangular matrix

Eg:- $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2×3 matrix

3. A matrix of order $1 \times m$ is called a row matrix

Eg:- $[1 \ 2 \ 3]_{1 \times 3}$

4. A matrix of order $n \times 1$ is called a column matrix

Eg:- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$

* Row and column matrices are also called as row and column vectors respectively

5. If $A = [a_{ij}]_{n \times n}$ such that $a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, then A is called a unit matrix. It is denoted by I_n.

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. If $A = [a_{ij}]_{m \times n}$ such that $a_{ij} = 0 \ \forall$ i and j then A is called zero matrix (or) a NULL matrix. It is denoted by '0' (or) more clearly $0_{m \times n}$

Eg: $0_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$

⇒ Diagonal element of a square matrix and Principal diagonal

Definition:-

1. In a matrix $A = [a_{ij}]_{n \times n}$, the elements a_{ij} of A for which $i = j$ (i.e., $a_{11}, a_{22}, \ldots a_{nn}$) are called the diagonal element of A. The line along which the diagonal elements lie is called the Principal diagonal of A.

2. A square matrix of all whose element except those in leading diagonal are zero is called diagonal matrix. If $d_1, d_2, \ldots d_n$ are diagonal element of a diagonal matrix A, then A is written as

$$A = diag(d_1, d_2, \ldots d_n)$$

Ex:- $A = diag(3, 1, -2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

3. A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

Ex:- $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Equal Matrix:-

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if

⇒ A & B are of the same type (or order)

⇒ $a_{ij} = b_{ij}$ for every i and j

Algebra of Matrices:

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ between be two matrix $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$ is called the sum of the matrices A and B the sum of A and B is called and denoted by A+B

Thus $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$ and

$[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

Difference of two Matrices

If A, B are two matrices of the same type (order) then $A+(-B)$ is taken as $A-B$

Multiplication of a Matrix by a Scalar

Let A be a matrix. The Matrix obtained by matrix multiplying every element of A by k by a scalar is called the product of A by k and is denoted by kA (or) Ak

Thus if $A = [a_{ij}]_{m\times n}$, then

$kA = [ka_{ij}]_{m\times n}$ and $[ka_{ij}]_{m\times n} = k[a_{ij}]_{m\times n} = kA$

Properties:

$\Rightarrow$ $0A = 0$ (null matrix), $(-1)A = -A$, called the negative of A.

$\Rightarrow$ $k_1(k_2A) = (k_1k_2)A = k_2(k_1A)$ where k_1, k_2 are scalars

$\Rightarrow$ $kA = 0 \Rightarrow A = 0$ if $k \neq 0$

iv) $k_1 A = k_2 A$ and A is not a null matrix $\Rightarrow k_1 = k_2$

Matrix Multiplication

Let $A = [a_{ik}]_{m \times n}$ and $B = [b_{kj}]_{n \times p}$, then the matrix $C = [c_{ij}]_{m \times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ is called the product of the matrices A and B in that order and we write $C = AB$

In the product AB, the matrix A is called the pre-factor and B the post-factor

If the number of coloumns of A is equal to the number of rows in B then the matrices are said to be comfortable for multiplication in that order

Positive Integral powers of square Matrices.

Let A be a square matrix then A^2 is defined as $A \cdot A$. Now, by the Associative law

$A^2 A = (A \cdot A) A = A(AA) = A \cdot A^2$ so that we can write

$$A^2 A = AA^2 = A \cdot A \cdot A = A^3$$

similarly we have $AA^{m-1} = A^{m-1} A = A^m$ where m is a positive integer

Furthur we have $A^m A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$ where m, n are positive integer

<u>Note:</u>

$$I^n = I \; ; \; O^n = O$$

Trace of a Square Matrix

Let $A = [a_{ij}]_{n \times m}$ then trace of the square matrix A is define as $\sum_{i=1}^{n} a_{ii}$ and is denoted by $tr(A)$

Thus $tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + --- + a_{nn}$

* Properties:

If A and B are square matrices of other n and l is any scalar, then

$\Rightarrow tr(lA) = l\,trA$

$\Rightarrow tr(A+B) = trA + trB$

$\Rightarrow tr(AB) = tr(BA)$

Triangular Matrix

A square matrix all of whose elements below the leading diagonal are zero is called an upper triangular matrix. A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix

Ex:. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ is an upper triangular matrix

and $\begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 & 0 \\ 2 & +1 & -8 & 5 & 0 \\ 2 & 0 & 4 & 1 & 6 \end{bmatrix}$ is an lower triangular matrix

⇒ If A is a square matrix such that $A^2 = A$ then A is called idempotent

⇒ If A is a square matrix such that $A^m = 0$ where M is a positive integer, then A is called Nilpotent. If M is least positive integer such that $A^m = 0$, then A is called Nilpotent of index M.

⇒ If A is a square matrix such that $A^2 = I$ then A is called involuntery

The transpose of a Matrix

Definition:

The matrix obtained from any given matrix A by inter changing its rows and columns is called the transpose of A. It is denoted by A' or A^T the transpose of A is.

If $A = [a_{ij}]_{m \times n}$, then the transpose of A is

$$A' = [b_{ji}]_{n \times m}, \text{ where } b_{ji} = a_{ij}$$

Also $(A')' = A$

Note:

If A' and B' be the transpose of A and B, respectively, then

⇒ $(A')' = A$

⇒ $(A+B)' = A' + B'$, A and B being of the

⇒ $(kA)' = kA'$, k is a scalar

⇒ $(AB)' = B'A'$, A & B being comformable for multiplication.

Determinants:

Minors and co-factors of a square Matrix

let $A = [a_{ij}]_{n \times m}$ be a square matrix. when from A the elements of i^{th} row and j^{th} column are deleted the determinant of (n-1) rows matrix M_{ij} is called the minor of a_{ij} of A and is denoted by $|M_{ij}|$. The signed minor $(-1)^{i+j}$ $|M_{ij}|$ is called the co-factor of a_{ij} and is denoted by A_{ij}

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Notes:

1. Determinant of the square matrix A can be defined as.

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

(or)

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

Therefore in a determinant the sum of the products of the elements of any row or column with their are corresponding co-factors is called to the value of the determinant

2. If A is a square matrix of order n then.
   $|KA| = K^n |A|$, where K is a scalar

3. If A is a square matrix of order n, then
   $|A| = |A^T|$

4. If A and B be two square matrices of the same order then $|AB| = |A| \cdot |B|$

* Adjoint of a Square Matrix

let A be a square matrix of order n The transpose of the matrix got from A by replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A

Note: For any scalar K, $adj(KA) = K^{n-1}\, adj A$

* Singular and Non-singular Matrices:

Definations:

A square matrix A is is said to be singular if $|A| = 0$ if $|A| \neq 0$, then A is said to be non-Singular. Thus only non-singular matrix possess inverses.

Note :-

If A, B are non - singular then AB, the product is also non-singular matrixes is also non singular

Inverse of a Matrix :-

let A be any square matrix B, it exists such that $AB = BA = I$, then B is called inverse of A and is denoted by A^{-1}

Note:-

For AB, BA to be both defined and equal it is necessary that A and B are both square matrices of same order. Thus a non - square matrix cannot have inverse.

Inventible

A matrix is said to be inverting, if it possess inverse.

Crammer's Rule (Determinant) Method

The solution of the system of linear equation

$a_1 x + b_1 y + c_1 z = d_1;$

$a_2 x + b_2 y + c_2 z = d_2;$

$a_3 x + b_3 y + c_3 z = d_3;$ is given by

$x = \frac{\Delta_1}{\Delta} \Rightarrow y = \frac{\Delta_2}{\Delta}; z = \frac{\Delta_3}{\Delta}$ $(\Delta \neq 0)$, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} ; \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} ; \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

we notice that $\Delta_1, \Delta_2 \cdot \Delta_3$ are the determinant obtained from Δ on replacing the 1st, 2nd and 3rd columns by d' values respectively

Symmetric Matrix :-

A square Matrix 'A' = $[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus, A is a symmetric matrix $\Leftrightarrow A = A'$ or

$A' = A$

Skew - Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be Skew Symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus A is a skew symmetric matrix $\Leftrightarrow A = A'$
$A' = A$

Note:

Every diagonal element of a skew-symmetric matrix is necessarily zero since

$$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$$

Ex :. $\begin{bmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{bmatrix}$ is a symmetric matrix

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$ is a skew - symmetric matrix

Properties :

1) * A is symmetric

* KA is symmetric

2) A is skew - symmetric

KA is skew- symmetric

Orthogonal Matrix

A square matrix 'A' is said to be orthogonal $AA' = A'A = I$, that is $A^T = A^{-1}$

Solved Examples

1. proved that $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal

Solu) $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$

$A^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9}+\frac{4}{9}+\frac{4}{9} & \frac{2}{9}+\frac{2}{9}-\frac{4}{4} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} \\ \frac{2}{9}+\frac{2}{9}-\frac{4}{9} & \frac{4}{9}+\frac{1}{9}+\frac{4}{9} & \frac{4}{9}-\frac{2}{9}-\frac{2}{9} \\ \frac{2}{9}+\frac{4}{9}+\frac{2}{9} & \frac{4}{9}-\frac{2}{9}-\frac{2}{9} & \frac{4}{9}+\frac{4}{9}+\frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$A \cdot A^T = I_3$$

∴ Given matrix is an orthogonal

2. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 9 \end{bmatrix}$ is orthogonal

Solu: let $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 9 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 9 \end{bmatrix} \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$$

Given matrix is not an orthogonal

3. Find the values of A, B and c when

$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$ is orthogonal

Solu) let $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 0 & 2b & c \\ a & b & c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

$$= \begin{bmatrix} 0+4b^2+c^2 & 0+2b^2-c^2 & 0-2b^2+c^2 \\ 0+2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ 0-2b^2+c^2 & a^2-b^2-c^2 & a^2-b^2+c^2 \end{bmatrix}$$

Given that $A \cdot A^T = I_3$

$$= \begin{bmatrix} 4b^2+c & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{aligned} &2b^2-c^2=0 \rightarrow (1) \\ &a^2-b^2-c^2=0 \rightarrow (2) \\ &4b^2+c^2=1 \rightarrow (3) \\ &a^2+b^2+c^2=1 \rightarrow (4) \end{aligned}$$

from (1) $2b^2-c^2=0$

$c^2 = 2b^2$

from (2) $a^2-b^2-c^2=0$ | $a^2 = 3b^2$

$a^2-b^2-2b^2=0$ | $a = \sqrt{3}b$

Rank of a Matrix

* If A is a null matrix we define its rank will be "zero". If A
* If A is a non zero matrix we say that R is the rank of A if the following conditions are satisfied

1. Every $(r+1)^{th}$ order minor of A is zero
2. There exsist atleast one rth order minor of A which is not zero
3. Rank of A is denoted by "$\rho(A)$"

Note:

* Every matrix will have a rank
* Rank of A matrix is unique
* Rank of A is ≥ 1 when A is a non-zero matrix
* If A is a matrix of order $m \times n$ then rank of A is $\leq \min(m, n)$.
* If rank of $A = r$ then every minor of A of order $(r+1)$ or more is zero.
* rank of the Identity matrix I_n is 'n'
* If A is matrix of order 'n' and A is non-singular $(|A| \neq 0)$ then rank of $A = n$.

1. Find the rank of the matrix

i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Solu) i) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$|A| = 1(48-40) - 2(36-28) + 3(30-28)$

$= 8 - 16 + 6$

$= -2 \neq 0$

$\therefore \rho(A) = 3$

ii) $|A| = \begin{vmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix}$

$|A| = 3(4-4) + 1(-12+12) + 2(-6+6)$

$|A| = 0$

$\rho(A) < 3$

A minor of order 2×2 of A is $\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 6-6 = 0$

" " " $\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = -4-4 = -8 \neq 0$

$\rho(A) = 2$

iii) H.W $\begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ v) $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

Solu) iii) Given matrix

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{vmatrix}$$

$$= -1(18-1) - 0(9+5) + 6(3+30)$$

$$= -1(17) - 0 + 6(33)$$

$$= -17 + 198$$

$$|A| = 181 \neq 0$$

$$\therefore \rho(A) = 3$$

iv) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 1(24-25) - 2(18-20) + 3(15-16)$$

$$= 1(-1) - 2(-2) + 3(-1)$$

$$= -1 + 4 - 3$$

$$|A| = 0$$

$$\rho(A) < 3$$

A minor of order 2×2 of A is $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$= 4 - 6 = -2 \neq 0$$

$$\rho(A) = 2$$

v) Given matrix

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 5 & 2 & 4 \end{vmatrix}$$

Here A minor of 3x3 of A is

$|A| = 2(16+4) + 1(4+10) + 3(2-20)$ $\rho(A) \leq \min(3,4)$

$\rho(A) \leq 3$

$= 2(20) + 1(14) + 3(-18)$

$= 40 + 14 - 54$

$|A| = 0$

A minor of order 3x3 of A is $\begin{vmatrix} -1 & 3 & 1 \\ 4 & -2 & 1 \\ 2 & 4 & 3 \end{vmatrix}$

$|A| = -1(-6-4) - 3(12-2) + 1(16+4)$

$= -1(-10) - 3(10) + 1(20)$

$= 10 - 30 + 20$

$|A| = 0$

A minor of order 3x3 of A is $\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ 5 & 4 & 3 \end{vmatrix}$

$|A| = 2(-6-4) - 3(3-5) + 1(4+10)$

$= 2(-10) - 3(-2) + 1(14)$

$= -20 + 6 + 14$

$= \cancel{-20+20}$

$|A| = 12 \cancel{-10} = 0$

$\rho(A) \neq 3$

$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 4 & 1 \\ 5 & 2 & 3 \end{vmatrix} = 2(12-2) + 1(3-5) + 1(2-20)$

$= 2(10) + 3(-2) + 1(-18)$

$= 20 - 6 - 18$

$= 0$

Now A minor of order 2×2 of A is

$$\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 8+1 = 9 \neq 0$$

$$\rho(A) = 2$$

13) From ③

$$4b^2 + c^2 = 1$$

$$4b^2 + 2b^2 = 1$$

$$6b^2 = 1$$

$$b^2 = \frac{1}{6}$$

$$b = \frac{1}{\sqrt{6}}$$

$$c^2 = 2b^2 = \cancel{2} \cdot \frac{1}{\cancel{6}\,3} = \frac{1}{3} \Rightarrow c = \frac{1}{\sqrt{3}}$$

$$a = \sqrt{3},\ b = \frac{1}{\sqrt{6}},\ c = \frac{1}{\sqrt{3}}$$

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Conjugate of the Matrix

The matrix obtained from any given matrix A are replacing its elements by the co-orresPonding Conjugate Complex Numbers is called the Conjugate of A. It is denoted by '$\bar{A}$'.

Ex: $A = \begin{bmatrix} 2+3i & 0 & i \\ i+2 & 2i-3 & 7 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2-3i & 0 & -i \\ -i+2 & -2i-3 & 7 \end{bmatrix}$$

Note:

1. If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then

* $\overline{(\bar{A})} = A$
* $\overline{(A \pm B)} = \bar{A} \pm \bar{B}$
* $\overline{(KA)} = \bar{K}\bar{A}$
* $\overline{(AB)} = \bar{A} \cdot \bar{B}$

The transpose of the conjugate of a square matrix

The transpose of a square matrix and its conjugate is

* If A is a square matrix $\bar{A}$ then the transpose of $\bar{A}$ is $(\bar{A})^T$
* The transposed conjugate of A is denoted by (transposed) "A^θ"
* Therefore $(\bar{A})^T = \overline{(A^T)} = A^\theta$

Ex:. $A = \begin{bmatrix} 5 & 3-i & -2i \\ 6 & 1+i & 4-i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 5 & 3+i & 2i \\ 6 & 1-i & 4+i \end{bmatrix}$

$(\bar{A})^T = \begin{bmatrix} 5 & 6 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix} = A^\theta$

Note

1. If A^θ and B^θ be the transposed conjugates of A and B respectively

* $(A^\theta)^\theta = A$
* $(A \pm B)^\theta = A^\theta \pm B^\theta$
* $(KA)^\theta = \bar{K}A^\theta$ where K is a complex number.
* $(AB)^\theta = B^\theta \cdot A^\theta$

Hermitian Matrix

A square matrix A such that $(\bar{A})^T = A$ is called a Hermitian Matrix

Ex: $A = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$$(\bar{A})^T = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} = A$$

$\therefore A$ is a Hermitian Matrix

Skew Hermitian Matrix

A square matrix A such that $(\bar{A})^T = -A$ is called a skew Hermitian Matrix

Ex:

$A = \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}, \bar{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$

$$(\bar{A})^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix} = -A$$

$\therefore A$ is a skew-Hermitian Matrix

Note:

1. It should be noted that elements are the leading diagonals must be all zero or all are purely imaginary

Unitary Matrix

A square matrix A such that $(\bar{A})^T = A^{-1}$ is called a unitary Matrix

$\therefore A^{\theta}.A = AA^{\theta} = I$

Date 28/11/18

1. If A and B are Hermitian matrices prove that AB − BA is a skew Hermitian Matrices

Solu) Given that A, B are Hermitian matrices

$$(\bar{A})^T = A, \quad (\bar{B})^T = B$$

$$(\overline{AB - BA})^T = (\overline{AB} - \overline{BA})^T$$

$$= (\overline{AB})^T - (\overline{BA})^T$$

$$= (\bar{A}\bar{B})^T - (\bar{B}\bar{A})^T$$

$$= (\bar{B})^T(\bar{A})^T - (\bar{A})^T(\bar{B})^T$$

$$= BA - AB$$

$$(\overline{AB - BA})^T = -(AB - BA)$$

∴ AB − BA is a skew-Hermitian Matrix

2. If A is a Hermitian matrix prove that iA is a skew Hermition matrix

solu) since A is a Hermitian matrix

$$(\bar{A})^T = A \Rightarrow A^\theta = A$$

$$(iA)^\theta = \bar{i}A^\theta$$

$$= -iA$$

$$(iA)^\theta = -(iA)$$

∴ iA is a skew-Hermitian matrix

3. If A is a skew Hermitian prove that iA is a Hermitian matrix

solu) Since A is a skew Hermitian matrix

$$A^\theta = -A$$

$$(iA)^\theta = \bar{i}A^\theta$$

$$= -i - A$$

$$(iA)^\theta = iA$$

$\therefore iA$ is Hermitian matrix

4. Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew Hermitian matrix

Solu) Since A is a square matrix

$$(A+A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A$$

$$(A+A^\theta)^\theta = A + A^\theta$$

$\therefore A+A^\theta$ is a Hermitian matrix

$\frac{1}{2}(A+A^\theta) = P$ is also a Hermitian matrix

Now $(A-A^\theta)^\theta = A^\theta - (A^\theta)^\theta$

$$= A^\theta - A$$

$$= -(A-A^\theta)$$

$\therefore (A-A^\theta)$ is a skew - Hermitian matrix

$\therefore \frac{1}{2}(A-A^\theta) = Q$ is also a skew Hermatian matrix

$$P+Q = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$$

$$= A$$

$\therefore$ A square matrix A is uniquely expressible a sum of Hermitian and skew Hermitian matrix.

5. If $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is a Hermition matrix and iA is a skew Hermition matrix

Solu) $\overline{A} = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ 2-5i & 3-i & 4 \end{bmatrix}$

$A^\theta = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ 2+5i & 3+i & 4 \end{bmatrix}$

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$(A+A^\theta)^\theta = A^\theta + (A^\theta)^\theta$

$= A^\theta + A$

$= A + A^\theta$

$\therefore A + \bar{A}^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ 2-5i & 3-i & 4 \end{bmatrix} + \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ 2+5i & 3+i & 4 \end{bmatrix}$

$A + A^\theta = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ 2-5i & 3-i & 4 \end{bmatrix} + \begin{bmatrix} 3 & 7-4i & 2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

Given matrix //

$A = \begin{bmatrix} 3 & 7-4i & -2+3i \\ 7+4i & -2 & 3+i \\ 2-5i & 3-i & 4 \end{bmatrix}$

$\bar{A}^T = \begin{bmatrix} 3 & 7+4i & -2-3i \\ 7-4i & -2 & 3-i \\ 2+5i & 3+i & 4 \end{bmatrix} \Rightarrow A^\theta = \begin{bmatrix} 3 & 7-4i & 2+5i \\ 7+4i & -2 & 3+i \\ -2-3i & 3-i & 4 \end{bmatrix}$

$(\bar{A})^T = A$

$iA = i \begin{bmatrix} 3 & 7-4i & -2+3i \\ 7+4i & -2 & 3+i \\ 2-5i & 3-i & 4 \end{bmatrix}$

wrong

$= \begin{bmatrix} 3i & 7i+4 & -2i-3 \\ 7i-4 & -2i & 3i-1 \\ 2i+5 & 3i+1 & 4i \end{bmatrix}$

$(\bar{iA}) = \begin{bmatrix} -3i & -7i+4 & +2i-3 \\ -7i-4 & 2i & -3i-1 \\ -2i+5 & -3i+1 & -4i \end{bmatrix}$

$(\bar{iA})^T = \begin{bmatrix} -3i & -7i-4 & -2i+5 \\ -7i+4 & 2i & -3i+1 \\ 2i-3 & -3i-1 & -4i \end{bmatrix} = -i \begin{bmatrix} 3 & 7+4 \\ & \end{bmatrix}$

solu) Given $\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & -3i \\ -2+5i & 3i & 4 \end{bmatrix}$

$(\bar{A})^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$

$\therefore A$ is a Hermitian matrix

$iA = \begin{bmatrix} 3i & 7i+4 & -2i-5 \\ 7i-4 & -2i & 3i-1 \\ -2i+5 & +3i+1 & 4i \end{bmatrix}$

$\bar{iA} = \begin{bmatrix} -3i & -7i+4 & 2i-5 \\ -7i-4 & 2i & -3i-1 \\ +2i+5 & -3i+1 & -4i \end{bmatrix}$

$(\bar{iA})^T = \begin{bmatrix} -3i & -7i-4 & 2i+5 \\ -7i+4 & 2i & -3i+1 \\ 2i-5 & -3i-1 & -4i \end{bmatrix}$

$= -i\begin{bmatrix} 3 & 7+4i & 2+5i \\ 7-4i & -2 & 3+i \\ -2+5i & 3+i & 4 \end{bmatrix}$

$(\bar{iA})^T = -iA$

Date 29/11/18

6. Express the matrix $\begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ as the sum of Hermitian matrix and skew Hermitian matrix

solu) Given matrix $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$

$(\bar{A})^T = A^\theta = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$

$$A + A^{\theta} = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix} + \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2-2i & 4-6i \\ 2i+2 & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^{\theta}) = \begin{bmatrix} 1 & 1-i & 2-3i \\ i+1 & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

P is a Hermitian matrix

$$A - A^{\theta} = \begin{bmatrix} 2i & 2+2i & 6-4i \\ 2i-2 & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^{\theta}) = \begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

Q is a skew Hermitian matrix

$$P+Q = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$$= \begin{bmatrix} 1 & 1-i & 2-3i \\ i+1 & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} + \begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$$

$\therefore$ A square matrix can be expressed in sum of Hermitian and skew Hermitian matrix

$= A$

7. Express the matrix $\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and skew Hermitian matrix

Solu) Given $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$(\bar{A})^T = A^\theta = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$(A+A^\theta) = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

$$P = \frac{1}{2}(A+A^\theta) = \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix}$$

P is a Hermitian matrix.

$$A - A^\theta = \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$Q = \frac{1}{2}(A-A^\theta) = \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$$

$\therefore$ Q is a skew Hermitian matrix

$$P+Q = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$$

$$= \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix} + \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$$

$$= \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$$

$= A$

A square matrix can be expressed in the sum of the Hermitian and skew Hermitian matrix

Date 30/11/2018

Echelon form of a Matrix

A matrix is said to be in Echelon form if it has the following properties

1. zero rows if any are below any non-zero row
2. The first non-zero entry in each non-zero row is equal to one.
3. The no. of zeroes before the first non-zero elements in a row is less than the no. of such zeroes in the next row.

Note: The condition 2 is optional

Important Note

The no. of non-zero rows in the row Echelon form of A is the rank of A.

Ex:. $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$ $\rho(B) = 3$ $\rho(C) = 2$

1. Reduce the Matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into Echelon form and hence find its rank

solu)

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$\therefore \ \rho(A) = 3$

2. Reduce the Matrix $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ into Echelon form and hence find its rank

solu)

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$= \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{array}{l} R_3 \to 2R_3 - 11R_2 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 2R_4 \\ R_3 \to R_3 + 6R_4 \end{array}$$

$$\sim \begin{bmatrix} -1 & -3 & 3 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\sim \begin{bmatrix} -1 & -3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \frac{R_2}{-2}\ ;\ R_3 \leftrightarrow R_4$$

H.W

3. $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ 6. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ 7. $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

Solu) 3. $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} R$

$$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \qquad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_3 \to R_3 - R_4 \end{array}$$

$$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & -1 & 0 & 2 \\ -2 & 0 & -2 & 0 \end{bmatrix} \qquad R_4 \to R_4 + R_1$$

$$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 2 & 2 & -2 \\ 0 & -2 & 2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_3 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_3 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$\rho(A) = 2$

4. $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ $\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -10 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \leftrightarrow R_4$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -10 & -14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_3 \end{matrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & -14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_3 \\ R_3 \to 2R_3 - R_4 \end{matrix}$$

$\rho(A) = 2$

5. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 8 & 1 & 3 & 6 \\ -24 & 0 & -7 & -16 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \qquad \rho(A) = 3$$

6. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ -5 & 0 & -8 & -1 \\ 8 & 0 & 20 & 4 \end{bmatrix}$ $R_2 \to 3R_2 - R_1$, $R_3 \to 3R_3 + R_1$

$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ -5 & 0 & -8 & -1 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ $R_3 \to \frac{R_3}{4}$ $\therefore \rho(A) = 3$

(1) $\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ $R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$\sim \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ R. $R_3 \to 3R_3 + R_2$, $R_4 \to 3R_4 \pm R_2$

$\rho(A) = 2$

7. $= \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & +3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 + R_1$

$\sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$ $R_2 \to R_2 - R_3$

$\sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 2 \end{bmatrix}$ $R_3 \to R_3 + R_4$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4 \qquad \rho(A) = 2$$

Date 1/12/2018 **Reduction to Normal form**

Every $m \times n$ - matrix of rank "r" can be reduced to the form is iR (or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite change of elementory row or coloumn operations this form is called normal form or first Cananical form of a matrix

1. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

solu) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix} \quad R_2 \to R_2 + 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \quad C_2 \to C_2 - 2C_1, \; C_3 \to C_3 - C_1, \; C_4/8$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & 4 \end{bmatrix} \quad R_3 \to 4R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 9 & 1 \end{bmatrix} \quad C_2/8, \; C_4/4.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 1 \end{bmatrix} \quad C_3 \to C_3 - 5C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_3/9$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_4 \to C_4 - C_3$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore \rho(A) = 3$$

2. $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ by Conanical form

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - 2R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to 3R_3 - 2R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 8 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_2 \to 2C_2 - C_1 \\ C_3 \to 2C_3 - 3C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_1/2 \\ C_2/6 \\ C_3/8 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 - C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore \rho(A) = 3$$

H.W

3. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Sol: Given matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & -7 & 9 & -1 \\ 0 & -6 & 3 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -10 & -3 & -7 \\ 0 & -14 & 9 & -1 \\ 0 & -12 & 3 & -4 \end{bmatrix} \quad \begin{matrix} C_2 \to 2C_2 - 3C_1 \\ C_3 \to 2C_3 + C_1 \\ C_4 \to 2C_4 + C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -7 \\ 0 & -7 & 3 & -1 \\ 0 & -6 & 1 & -4 \end{bmatrix} \quad \begin{matrix} C_1/2 \\ C_2/2 \\ C_3/3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -7 \\ 0 & -22 & 3 & -1 \\ 0 & -11 & 1 & 4 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 5C_3 \\ C_3 \to \end{matrix}$$

-6 -5

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -7 \\ 0 & -22 & 0 & -22 \\ 0 & -11 & 1 & 4 \end{bmatrix} \quad R_3 \to R_3 + 3R_2, \; R_4 \to 2R_4 - 1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -7 \\ 0 & -22 & 0 & -22 \\ 0 & 0 & 2 & 18 \end{bmatrix} \quad R_4 \to 2R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -7 \\ 0 & -2 & 0 & 22 \\ 0 & 0 & 2 & 18 \end{bmatrix} \quad C_2/11, \; C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \\ 0 & 1 & -22 & 22 \\ 0 & 0 & -4 & 18 \end{bmatrix} \quad C_2/2, \; C_3 \to 7C_3 \pm C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 1 & 22 & -22 \\ 0 & 0 & 18 & -4 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 8 & -4 \end{bmatrix} \quad C_3 \to C_3 + C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & +11 \\ 0 & 0 & 8 & 2 \end{bmatrix} \quad C_4 \to C_4/2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 8 & 2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \quad \frac{R_3}{-7}, \; \frac{R_4}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to R_4 - 4R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad C_4 \to C_4 - 11C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \leftrightarrow R_3 \sim \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 4$$

4- $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 3 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -3 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

7- $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ 9. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ 11. $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

solu/ 11.

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & -13 & -13 \\ 0 & -9 & -9 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{R_2}{-13} \\ R_3 \to \frac{R_3}{-9} \end{array}$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 - 3R_3$$

$$\sim \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_1 \to R_1 - 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_1 \to R_1/2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_2 \to C_2 - 2C_1$$

4 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 2 \\ 0 & -6 & -4 & -22 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 + 3R_1 \\ R_3 \to R_3 + 3P \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 2 \\ 0 & 3 & 2 & 11 \end{bmatrix} \quad R_3 \to \frac{R_3}{-2}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 10 & 2 \\ 0 & 3 & -5 & 5 \end{bmatrix} \quad \begin{array}{l} C_3 \to 2C_3 - 3C_2 \\ C_4 \to C_4 - 2C_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -1 & 5 \end{bmatrix} \quad C_2 \to \frac{C_3}{5}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & -6 & 5 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - C_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{bmatrix} \quad \frac{C_2}{3}, \frac{C_3}{-2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - C_3 \\ C_4 \to C_4 - 5C_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \frac{C_2}{2} \quad \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

5 $$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 2 \\ 0 & -4 & -11 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & -2 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to 2R_1 + R_3 \\ R_4 \to R_4 - R_3 \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to \frac{R_1}{2} \\ R_3 \to \frac{R_3}{2} \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad \begin{array}{l} C_2 \to \frac{C_2}{-2} \\ C_3 \to C_3 + C_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - R_4 \\ R_3 \to 2R_3 - 3R_4 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \begin{matrix} R_4 \to 8R_1 \cdot \; C_2 \to \frac{C_2}{2} \\ C_4 \to C_4 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad C_4 \to C_4 + C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad C_4 \to \frac{C_4}{3}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + R_2$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 3$$

6. $$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -3 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -5 \\ 0 & -7 & 9 & -1 \\ 0 & -6 & 3 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & +1 \\ 0 & -14 & -3 & +5 \\ 0 & 20 & 9 & +1 \\ 0 & 3 & 3 & +4 \end{bmatrix} \quad C_2 \to C_2 + 3C_3, \quad C_4 \to \frac{C_4}{-1}$$

$$\sim \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & -14 & -3 & 5 \\ 0 & 20 & 9 & 1 \\ 0 & 3 & 38 & 4 \end{bmatrix} \quad \frac{C_1}{2} \quad R_4 \to {}^{+14}R_4 + 3R_2$$

$+14\,(0 \;\; 3 \;\; 3 \;\; 4)$
$0 \;\; +42 \;\; +42 \;\; +56$
$3\,(0 \;\; -14 \;\; -3 \;\; 5)$
$0 \;\; -42 \;\; -9 \;\; 15$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -14 & -3 & 5 \\ 0 & 20 & 9 & 1 \\ 0 & 3 & 3 & 4 \end{bmatrix} \quad C_3 \to -C_3 + C_1, \quad C_4 \to C_4 - C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -14 & -1 & 5 \\ 0 & 20 & 3 & 1 \\ 0 & 3 & 1 & 4 \end{bmatrix} \quad C_3 \to \frac{C_3}{3}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & -22 & 3 & 1 \\ 0 & -11 & 1 & 4 \end{bmatrix} \quad C_2 \to C_2 - 14C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 4 \end{bmatrix} \quad C_2 \to \frac{C_2}{-11}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 1 & 1 & 4 \end{bmatrix} \quad R_3 \to R_3 - 2R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -7 \\ 0 & 1 & 0 & 11 \end{bmatrix} \quad R_2 \to R_2 + R_3, \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -7 & 1 & 0 \\ 0 & 11 & 0 & 1 \end{bmatrix} \quad C_2 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 1 \end{bmatrix} \quad R_3 \to 2R_3 - 7R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 1 \end{bmatrix} \quad R_2 \to \frac{R_2}{-2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to R_4 - 11R_2$$

$$\sim \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 4$$

7. $$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 + 7R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -5 \end{bmatrix} \quad R_4 \to \frac{R_4}{6}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_3 \\ R_4 \to R_4 - R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad \begin{array}{l} C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 6C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad C_4 \to C_4 + 9C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad C_4 \to C_4 + 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to \frac{R_4}{-3}$$

$$\therefore \rho(A) = 4$$

8 $$A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ -1 & 1 & 2 & 2 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to \frac{R_4}{3} \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 5 & 5 & 0 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{R_2}{5} \\ R_3 \to \frac{R_3}{10} \\ R_4 \to R_4 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 4R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 5R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 2C_1 \\ C_5 \to C_5 + 3C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_3 \to C_3 - C_2 \\ C_5 \to C_5 - C_2 \end{array}$$

when $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore \rho(A) = 2$

9 $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_3 \to R_3 - 3R.$

$\sim \begin{bmatrix} 1 & 2 & -5 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \to R_1 + R_4$

$\sim \begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & -2 & 6 & 2 \\ 0 & -5 & 15 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \end{bmatrix}$ $R_1 \to R_1 + R_2$, $R_3 \to \frac{R_3}{5}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & +2 & 6 & 2 \\ 0 & +1 & 3 & 1 \\ 0 & +1 & 3 & 1 \end{bmatrix}$ $C_2 \to \frac{C_2}{-1}$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $C_3 \to \frac{C_3}{3}.$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $R_2 \to \frac{R_2}{2}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $C_3 \to C_3 - C_2$, $C_4 \to C_4 - C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

Date
3/12/2018 System of Linear Simultaneous equations

1. Write the following equations in matrix form Ax = B and solve for x by finding A^{-1} where $Ax = B$ and solve for x by finding A^{-1}

$x + y - 2z = 3$; $2x - y + z = 0$; $3x + y - z = 8$

Solu) Given Equations

$$x + y - 2z = 3$$
$$2x - y + z = 0$$
$$3x + y - z = 8$$

$AX = B \Rightarrow X = A^{-1}B$

when $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$

consider $A = I_3 A$.

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 5 \\ 0 & -2 & 5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A.$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 5 \\ 0 & 0 & 5 \end{bmatrix} R_3 \to 3R_3 - 2R_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & -2 & 3 \end{bmatrix} A$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to 3R_1 + R_2 \\ R_2 \to R_2 - R_3 \\ R_3/5 \end{matrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & -3 \\ -1 & -2/5 & 3/5 \end{bmatrix} A$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 / -3 \end{matrix} = \begin{bmatrix} 0 & 3/5 & 3/5 \\ -1 & -1 & 1 \\ -1 & -\frac{2}{5} & 3/5 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{R_1}{3}} = \begin{bmatrix} 0 & \frac{1}{5} & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} A$$

$$I_3 = CA. \Rightarrow C = \begin{bmatrix} 0 & \frac{1}{5} & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix}.$$

$$\boxed{\therefore AA^{-1} = A^{-1}A = I}$$

$$\therefore A^{-1} = C$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+8/5 \\ -3-0+8 \\ -3-0+24/5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/5 \\ 5 \\ 9/5 \end{bmatrix}$$

Date
4/12/2018

$\therefore x = 8/5,\ y = 5,\ z = 9/5$

2. For Non-Homogeneous System

* Consistent

The system $AX = B$ is consistent if and only if it has a solution rank of A = rank of AB and

1. The $\rho(A) = \rho(AB) = n$ then the system has unique solution.
   where n = unknown variables

2. If $\rho(A) = \rho(AB) < n$ then the system is consistent but therese exsist infinite number of solutions.

3. If the $\rho(A) \neq \rho(AB)$ then the system is inconsistent and it has no solution.

1. Show that the equations $x+y+z=4$; $2x+5y-2z=3$ and $x+7y-7z=5$ are not consistent (only row operations)

Solu) Given Equations

$$x+y+z=4$$
$$2x+5y-2z=3$$
$$x+7y-7z=5$$

can be expressed as $AX=B$

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & -2\\ 1 & 7 & -7\end{bmatrix} ; B=\begin{bmatrix}4\\3\\5\end{bmatrix} ; X=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Consider Argumented matrix

$$[AB]=\begin{bmatrix}1 & 1 & 1 & 4\\ 2 & 5 & -2 & 3\\ 1 & 7 & -7 & 5\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 1 & 1 & 4\\ 0 & 3 & -4 & -5\\ 0 & 6 & -8 & 1\end{bmatrix} \quad \begin{matrix}R_2 \to R_2-2R_1\\ R_3 \to R_3-R_1\end{matrix}$$

$$\sim \begin{bmatrix}1 & 1 & 1 & 4\\ 0 & 3 & -4 & -5\\ 0 & 0 & 0 & 11\end{bmatrix} \quad R_3 \to R_3-2R_2$$

$\rho(A)=2$; $\rho(AB)=3$

$\therefore \rho(A) \neq \rho(AB)$

Hence given equation are inconsistent and it has no solution

2. Solve the Equations $x+y+z=9$, $2x+5y+7z=52$ and $2x+y-z=0$

Solu) Given equations

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

Given equations can be expressed as

$$A=\begin{bmatrix}1&1&1\\2&5&7\\2&1&-1\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix};\ B=\begin{bmatrix}9\\52\\0\end{bmatrix}$$

$$\begin{array}{r}52\\18\\\hline 34\\34\\\hline -10\end{array}$$

Argumented matrix

$$[AB]=\begin{bmatrix}1&1&1&9\\2&5&7&52\\2&1&-1&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&9\\0&3&5&34\\0&-1&-3&-18\end{bmatrix}\begin{array}{l}\\R_2\to R_2-2R_1\\R_3\to R_3-2R_1\end{array}$$

$$\sim\begin{bmatrix}1&1&1&9\\0&3&5&34\\0&0&-4&-20\end{bmatrix}\begin{array}{l}\\\\R_3\to 3R_3+R_2\end{array}$$

$$\rho(A)=3,\ \rho(AB)=3,\ n=3$$

$$\therefore\ \rho(A)=\rho(AB)=n$$

Given system is consistent and it has unique no. of so solutions

$$x+y+z=9$$
$$3y+5z=34\ \to ①$$
$$-4z=-20$$
$$z=5$$

$$① \Rightarrow\ 3y+25=34$$
$$3y=34-25$$
$$=9$$

$y = 3$

$x + 3 + 5 = 9$

$x = 1$

$\therefore x = 1, y = 3, z = 5$

3\. Solve the system of linear equations by matrix method $x + y + z = 6$; $2x + 3y - 2z = 2$; $5x + y + 2z = 13$

solu) Given equations

$x + y + z = 6$

$2x + 3y - 2z = 2$

$5x + y + 2z = 13$

~~Argum~~

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 6 \\ 2 \\ 13 \end{bmatrix}$$

Now $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix} = \begin{bmatrix} 6 \\ -10 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 + 4R_2 = \begin{bmatrix} 6 \\ -10 \\ -57 \end{bmatrix} \quad \frac{-17}{-40}$$

$x + y + z = 6$

$y - 4z = -10$

$-19z = -57$

$z = 3$

$y - 12 = -10$

$y = 2$

$x + 2 + 3 = 6$

$x = 1$

4. Examine the following equations are consistent or inconsistent

1) $x - 4y + 7z = 8$
$3x + 8y - 2z = 6$
$7x - 8y + 26z = 31$

2) $x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$
$x - y + z = -1$

solu/ 1) Given equations

$x - 4y + 7z = 8$
$3x + 8y - 2z = 6$
$7x - 8y + 26z = 31$

Can be expressed as

$$A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 8 \\ 6 \\ 31 \end{bmatrix}$$

Consider Argumented matrix $[AB]$

$$[AB] = \begin{bmatrix} 1 & -4 & 7 & 8 \\ 3 & 8 & -2 & 6 \\ 7 & -8 & 26 & 31 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 8 \\ 0 & 20 & -23 & -18 \\ 0 & 20 & -23 & -25 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 8 \\ 0 & 20 & -23 & -18 \\ 0 & 0 & 0 & -7 \end{bmatrix} R_3 \to R_3 - R_2$$

$\rho(A) = 2$; $\rho(AB) = 3$; $n = 3$

$\rho(A) \neq \rho(AB)$

Hence given equations are inconsistent and it has no solution

2) Given equations

$x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$
$x - y + z = -1$

Given equations can be expressed as

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

consider argumented matrix

$$[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{bmatrix} \quad \begin{array}{l} R_3 \to 7R_3 - 6R_2 \\ R_4 \to 7R_4 - 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 5 & 20 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$e(A) = \cancel{4}3 \quad ; \quad e(AB) = 4 \quad ; \quad n = 4$

$$e(A) \neq e(AB) = n$$

$\therefore$ The given system is ~~in~~consistent and it has unique solution

$$x + 2y - z = 3$$
$$-7y + 5z = -8$$
$$-z = -4$$
$$5z = 20$$

$-7y+5(4)=-8$

$-7y+20=-8$

$-7y=-28$

$y=4$

$x+2(4)-4=3$

$x+4=3$

$x=-1$

$x+2y-z=3$
$-1+2(4)-4=3$
$-5+8=3$
$3=3$

Date
5/12/2018

5. For what values of λ the equations $x+y+z=1$; $x+2y+4z=\lambda$; $x+4y+10z=\lambda^2$ have a solution and solve them completely in each case.

Solu) Given equation

$$\left.\begin{aligned} x+y+z&=1\\ x+2y+4z&=\lambda\\ x+4y+10z&=\lambda^2 \end{aligned}\right\} \rightarrow (1)$$

can be expressed as a matrix form of system (1)

$AX=B$ where

$$A=\begin{bmatrix}1&1&1\\1&2&4\\1&4&10\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix};\ B=\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}$$

Argumented matrix

$$[AB]=\begin{bmatrix}1&1&1&1\\1&2&4&\lambda\\1&4&10&\lambda^2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&3&9&\lambda^2-1\end{bmatrix}\ \begin{matrix}R_2\rightarrow R_2-R_1\\R_3\rightarrow R_3-R_1\end{matrix}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&0&0&\lambda^2-3\lambda+2\end{bmatrix}\rightarrow (2)\ \ R_3\rightarrow R_3-3R_2$$

$\rho(A) = \rho(AB) = 3$

But given that the system has a solution. It must be consistent. So that

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1) = 0$$

$$\lambda = 1, 2$$

Case (i)

if $\lambda = 1$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$; $\rho(AB) = 2$, $n = 3$

$\therefore \rho(A) = \rho(AB) = r < n$

Given equation are consistent and will have infinite no.of solutions.

$$x + y + z = 1$$

$$y + 3z = 0$$

let $n - r = 3 - 2 = 1$ L.I.S

let $z = k$

$$y + 3k = 0$$

$$y = -3k$$

$$x - 3k + k = 1$$

$$x = 1 + 2k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2k \\ -3k \\ k \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

case (ii)

if $\lambda = 2$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$; $\rho(AB) = 2$, $n = 3$

$\therefore \rho(A) = \rho(AB) = r < n$

Given equation consistent and will have infinite no. of solutions

$$x + y + z = 1$$

$$y + 3z = 1$$

let $z = k$

$$y + 3k = 1$$

$$y = 1 - 3k$$

$$x + \cancel{1} - 3k + k = \cancel{1}$$

$$x = 2k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ 1-3k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

2 If $a+b+c \neq 0$, show that the system of equation $-2x+y+z = a$, $x-2y+z = b$, $x+y-2z = c$ has no solution. If $a+b+c=0$. show that it has infinitely many solutions. show that it.

Solu) Given Equations

$$\left.\begin{aligned} -2x+y+z &= a \\ x-2y+z &= b \\ x+y-2z &= c \end{aligned}\right\} \rightarrow (1)$$

System (1) can be expressed as the matrix form of $AX = B$

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} ; \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Argumented matrix

$$[AB] = \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 3 & -3 & a+2c \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 0 & 0 & a+b+c \end{bmatrix} \quad R_3 \to R_3 + R_2$$

if $a+b+c \neq 0$

$\rho(A) = 2$; $\rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

Given system are inconsistent and will have no solution.

if $a+b+c = 0$

$\rho(A) = 2$; $\rho(AB) = 2$; $n = 3$

$\rho(A) = \rho(AB) = r < n$

Given equations are consistent and will have infinite no. of solutions

$x + y - 2z = c$

$-3y + 3z = b - c$

$n-r = 3-2 = 1$ L.I.S

let $z = k$

$-3y + 3k = b - c$

$3y = 3k - b + c$

$y = k - \frac{b}{3} + \frac{c}{3}$

$x + k - \frac{b}{3} + \frac{c}{3} - 2k = c$

$x = k + \frac{b}{3} + \frac{2c}{3}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k + \frac{b}{3} + \frac{2c}{3} \\ k - b/3 + c/3 \\ k \end{bmatrix}$$

$$= k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} b/3 + 2c/3 \\ -b/3 + c/3 \\ 0 \end{bmatrix}$$

H.W

3. Solve the system of linear equations by matrix method

i) over $x+y+z = 6$
$2x+3y-2z = 2$
$5x+y+2z = 13$

ii) $x+y+4z = 6$
$x+2y-2z = 6$
$x+y+z = 6$

iii) $x+y+2z = 4$
$2x-y+3z = 9$
$3x-y-z = 2$

iv) $x+y+z = 6$
$x+2y+3z = 14$
$x+4y+7z = 30$

iv)
Sol: Given Equations Matrix Method

$x+y+z = 6$

$x+2y+3z = 14$

$x+4y+7z = 30$ These equations can be expressed

$AX = B$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 6 \\ 8 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 - 3R_2 = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$x + y + z = 6$

$y + 2z = 8$ let $z = 0$

$y + 2(0) = 8$

$y = 8$

$x + 8 + 0 = 6$

$x = 6 - 8$

$x = -2$

$\therefore x = -2$; $y = 8$; $z = 0$

(or)

Consistent Method

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

Argumented matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$\therefore \rho(A) = 2 \;;\; \rho(B) = 2 \;;\; n = 3$

$\rho(A) = \rho(AB) < n$

The given system of equations is consistent and has infinite no. of solutions

$x+y+z = 6$;

$y+2z = 8$;

let $z = K$

$y+2K = 8$

$y = 8-2K$;

$\cancel{8}-\cancel{2K}+\cancel{2K} = \cancel{8}$

$x+8-2K+K = 6$

$x-K = 6-8$

$x-K = -2$

$x = -2+K$.

$x = K-2$

iii) Matrix Method

Given equations

$x+y+2z = 4$

$2x-y+3z = 9$

$3x-y-z = 2$

these equations can be expressed as $AX = B$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} ; \; B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix} ; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & -4 & -7 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} = \begin{bmatrix} 4 \\ 1 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & -4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to 3R_3 - 4R_2 = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$x + y + 2z = 4$

$-3y - z = 1$

$-17z = -34$

$z = 2$

$-3y - 2 = 1$

$-3y = 3$

$y = -1$

$x - 1 + 4 = 4$

$x = 1$

$\left[\because x + y + 2z = 4\right.$

$1 - 1 + 4 = 4$

$\left.0 = 0\right]$

$\therefore x = 1, \; y = -1, \; z = 2$

(or)

<u>Consistent</u> <u>Method</u>

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

Argumented matrix

$[A\,B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & 1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix} \quad R_3 \to 3R_3 - 4R_2$$

$\rho(A) = 3$; $\rho(AB) = 3$; $n = 3$

$\rho(A) \not= \rho(AB) = n$

the given system of equation is ~~in~~consistent

it has ~~a~~ a solution.

$$x + y + 2z = 4 \qquad ; \quad -3y - 2 = 1 \quad ; \quad x - 1 + 4 = 4$$
$$-3y - z = 1 \qquad\qquad -3y = 3 \qquad\qquad x = 1$$
$$-17z = -34 \qquad\qquad y = -1$$
$$z = 2$$

$\therefore x = 1$; $y = -1$; $z = 2$

ii) Given Equations

$$x + y + 4z = 6$$
$$x + 2y - 2z = 6$$
$$x + y + z = 6$$

$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$

<u>Matrix method</u>

The given system of equations can be expressed as $AX = B$

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + 4z = 6 \qquad ; \quad y = 0 \quad ; \quad x = 6 \quad ; \quad z = 0$$
$$y - 6z = 0$$
$$-3z = 0$$
$$z = 0$$

Consistent method

Argumented matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 1 & 2 & -2 & 6 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$\rho(A) = 3$; $\rho(AB) = 3$; $n = 3$

$\therefore$ $\rho(A) = \rho(AB) = n$

The given system of equations is consistent and has unique solution

$x + y + 4z = 6$; $y = 0$; $x + 0 + 0 = 6$

$y - 6z = 0$; $x = 6$

$-3z = 0$

$z = 0$

Date 6/12/18

4: Find the values of λ for which the system of equations $3x - y + 4z = 3$; $x + 2y - 3z = -2$; $6x + 5y + \lambda z = -3$ will have infinite no of solutions solve them with the λ values

Solu) Given equations

$$\left.\begin{aligned} 3x - y + 4z &= 3 \\ x + 2y - 3z &= -2 \\ 6x + 5y + \lambda z &= -3 \end{aligned}\right\} \rightarrow (1)$$

system (1) can be expressed in a matrix form

$AX = B$

where $A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$

$$[AB] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda-8 & -9 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

If $\lambda+5=0$

$\lambda = -5$

$\rho(A) = 2$; $\rho(AB) = 2$; $n=3$

$\rho(A) = \rho(AB) = r < n$

Given Equations have infinite no.of solutions

$\lambda = -5$ then

$$[AB] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n-r = 3-2 = 1$ L.I.S

$3x - y + 4z = 3$

$7y - 13z = -9$

let $z = k$

$7y - 13k = -9$

$7y = -9 + 13k$

$y = \frac{-9}{7} + \frac{13}{7}k$

$3x + \frac{9}{7} - \frac{13}{7}k + 4k = 3$

$3x = -\frac{15}{7}k + \frac{12}{7}$

$x = \frac{4}{7} - \frac{5}{7}k$

$\therefore x = \frac{4}{7} - \frac{5}{7}k$; $y = \frac{-9}{7} + \frac{13}{7}k$, $z = k$

5. Find whether the following set of equations are consistent

$x_1+x_2+x_3+x_4=0$

$x_1+x_2+x_3-x_4=4$

$x_1+x_2-x_3+x_4=-4$

$x_1-x_2+x_3+x_4=2$

Solu| Given equations

$$\left.\begin{array}{l} x_1+x_2+x_3+x_4=0 \\ x_1+x_2+x_3-x_4=4 \\ x_1+x_2-x_3+x_4=-4 \\ x_1-x_2+x_3+x_4=2 \end{array}\right\} \rightarrow ①$$

set ① can be expressend in a matrix form

$AX=B$ where

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} ; X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B=\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

Argumented matrix

$$[AB] \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & -2 & 0 & -4 \\ 0 & -2 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2-R_1 \\ R_3 \rightarrow R_3-R_1 \\ R_4 \rightarrow R_4-R_1 \end{array}$$

$\rho(A)=4, \ \rho(AB)=4; \ n=4$

$\rho(A)=\rho(AB)=r=n$

Given equations are consistent and will have a unique solution

Consistency of system of Homogeneous linear Equations

1. Consider a system of m-homogeneous linear equations in n-unknowns.

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n=0 \\ 2.\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n=0 \\ 3.\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\cdots+a_{3n}x_n=0 \\ 4.\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0 \end{array}\right\} \rightarrow (1)$$

system one can be return as in a matrix form

$$AX=0.$$

* If $\rho(A)=n$ then the system of equations have only trivial solution i.e., zero solution
* If $\rho(A)<n$, then the system of equations have an infinite no of non-trivial solutions, in this case $n-r$ linearly independent solution

1. Solve $x+y-2z+3w=0$; $x-2y+z-w=0$; $4x+y-5z+8w=0$; $5x-7y+2z-w=0$; Given equation

solu) Given equation

$$\left.\begin{array}{l} x+y-2z+3w=0 \\ x-2y+z-w=0 \\ 4x+y-5z+8w=0 \\ 5x-7y+2z-w=0 \end{array}\right\} \rightarrow (1)$$

system (1) can be expressed in the form of a matrix

$$AX=0$$

$$A=\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix};\ X=\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix};\ O=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & -12 & 12 & -16 & 0 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -3 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 4R_2 \end{array}$$

$\rho(A) = 2, \quad n = 4$

$r < n$

Given Equation have infinite no. of solutions of non-trivial solution.

$n - r = 4 - 2 = 2$ L.I.S

$x + y - 2z + 3w = 0$

$-3y + 3z - 4w = 0$

let $w = k_1, \quad z = k_2$

$3y = 3k_2 - 4k_1$

$y = k_2 - \frac{4}{3}k_1$

$x + k_2 - \frac{4}{3}k_1 - 2k_2 + 3k_1 = 0$

$x - k_2 + \frac{5}{3}k_1 = 0$

$x = k_2 - \frac{5}{3}k_1$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} k_2 - \frac{5}{3}k_1 \\ k_2 - \frac{4}{3}k_1 \\ k_2 \\ k_1 \end{bmatrix}$$

2. Solve $x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$

Solu) Given Equations

$$\left.\begin{aligned} x+y-3z+2w&=0\\ 2x-y+2z-3w&=0\\ 3x-2y+z-4w&=0\\ -4x+y-3z+w&=0\end{aligned}\right\} \rightarrow (1)$$

Given system of equations (1) can be expressed as $AX=0$

$$A=\begin{bmatrix}1 & 1 & -3 & 2\\ 2 & -1 & 2 & -3\\ 3 & -2 & 1 & -4\\ -4 & 1 & -3 & 1\end{bmatrix};\quad X=\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix};$$

$$\sim=\begin{bmatrix}1 & 1 & -3 & 2\\ 0 & -3 & 8 & -7\\ 0 & -5 & 10 & -10\\ 0 & 5 & -15 & 9\end{bmatrix}\begin{matrix}\\ R_2\rightarrow R_2-2R_1\\ R_3\rightarrow R_3-3R_1\\ R_4\rightarrow R_4+4R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & -3 & 2\\ 0 & -3 & 8 & -7\\ 0 & 0 & -10 & 5\\ 0 & 0 & -6 & -8\end{bmatrix}\begin{matrix}\\ \\ R_3\rightarrow 3R_3-5R_2\\ R_4\rightarrow 3R_4+5R_2\end{matrix}$$

$$\begin{array}{r}-35\\ 27\\ \hline -8\end{array}$$

$$\sim\begin{bmatrix}1 & 1 & -3 & 2\\ 0 & -3 & 8 & -7\\ 0 & 0 & -10 & 5\\ 0 & 0 & 0 & -21\end{bmatrix} R_4\rightarrow 2R_4-R_3$$

$\therefore \rho(A) = 4, \; n = 4$

$r = n$

Given Equation have trivial Solution

$x = 0 ; y = 0 ; z = 0 ; w = 0$

3 solve $x + y + w = 0 ; \; y + z = 0 ; \; x + y + z + w = 0 ;$
$x + y + 2z = 0$

Solu) Given equations

$$\left.\begin{aligned} x + y + w &= 0 \\ y + z &= 0 \\ x + y + z + w &= 0 \\ x + y + 2z &= 0 \end{aligned}\right\} \rightarrow (1)$$

system of Equation (1) can have be expressed in the form $AX = O$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix} \quad \begin{aligned} R_3 &\rightarrow R_3 - R_1 \\ R_4 &\rightarrow R_4 - R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad R_4 \rightarrow R_4 - 2R_3$$

$\sim \rho(A) = 4 \quad n = 4$

$\therefore$ The given system of equations can have trivial solution.

$x = 0 ; \; y = 0 ; \; z = 0 ; \; w = 0$

4. solve the system of equations $x+2y+(2+k)z=0$, $2x+(2+k)y+4z=0$, $7x+13y+(18+k)z=0$ for all values of k

Date 7/12/18

Solu) Given Equations

$$\left.\begin{aligned} x+2y+(2+k)z &= 0 \\ 2x+(2+k)y+4z &= 0 \\ 7x+13y+(18+k)z &= 0 \end{aligned}\right\} \rightarrow (1)$$

system (1) can be expressed as a matrix form of $AX=B$ where

$$A = \begin{bmatrix} 1 & 2 & 2+k \\ 2 & 2+k & 4 \\ 7 & 13 & 18+k \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The given system has a solution for all values of k if the system has a non-trivial solution. i.e.,

$$\rho(A) < n \quad ; n=3$$

$$\rho(A) < 3$$

Given matrix A is 3x3 matrix so that

$$|A| = 0$$

$$|A| = \begin{vmatrix} 1 & 2 & 2+k \\ 2 & 2+k & 4 \\ 7 & 13 & 18+k \end{vmatrix} = 0$$

$$1[(18+k)(2+k)-52] - 2[2(18+k)-28] + (2+k)[26-7(2+k)] = 0$$

$$36+18k+2k+k^2-52-2(36+2k-28)+(2+k)(26-14-7k)=0$$

$$k^2+20k-16-16-4k+24-14k+12k-7k^2=0$$

$$-6k^2+14k-8=0$$

$$3k^2-7k+4=0$$

$$3k^2-3k-4k+4=0$$

$$3k(k-1)-4(k-1)=0$$

$$(k-1)(3k-4)=0$$

12 → −4, −3

$(k-1)(3k-4)=0$

$k=1;\ k=4/3$

case(i)

If $k=1$

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 7 & 13 & 19\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & -1 & -2\end{bmatrix}\quad \begin{array}{l}R_2\to R_2-2R_1\\ R_3\to R_3-7R_1\end{array}$$

$$\sim\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 0 & 0\end{bmatrix}\quad R_3\to R_3-R_2$$

$\rho(A)=2,\ n=3,\ \rho(A)<n$

When $k=1$ the system has a non-trivial solution

$n-r=3-2=1$ L.I.S

$x+2y+3z=0$

$-y-2z=0$

Let $z=k$

$y=-2k$

$x-4k+3k=0$

$x=k$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}k\\ -2k\\ k\end{bmatrix}=k\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

case(ii)

If $k=4/3$

$$A=\begin{bmatrix}1 & 2 & 10/3\\ 2 & 10/3 & 4\\ 7 & 13 & 58/3\end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 10/3 \\ 0 & -2/3 & -8/3 \\ 0 & -1 & -12/3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 10/3 \\ 0 & -2/3 & -8/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to \frac{2}{3} R_3 - R_2$$

$$\rho(A) = 2; \; n = 3$$

$$\therefore \rho(A) < n$$

$n - r = 3 - 2 = 1$ L.I.S

$x + 2y + 10/3 z = 0, \quad -\frac{2}{3} y - \frac{8}{3} z = 0$

let $z = k$

$-\frac{2}{3} y - \frac{8}{3} k = 0 \; ; \quad y = \frac{\cancel{8}^{4}}{\cancel{3}} k \times -\frac{\cancel{3}}{\cancel{2}}$

$-\frac{2}{3} y = \frac{8}{3} k \; ; \quad y = -4k$

$x - 8k + \frac{10}{3} k = 0$

$x = \frac{14}{3} k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{14}{3} k \\ -4k \\ k \end{bmatrix} = k \begin{bmatrix} 14/3 \\ -4 \\ 1 \end{bmatrix}$$

-24
$\underline{10}$
$-\frac{14}{3}$

5. Solve the system $\lambda x + y + z = 0$; $x + \lambda y + z = 0$; $x + y + \lambda z = 0$; if it has non-zero solutions only

<u>solu</u> Given Equations

$$\left.\begin{array}{l} \lambda x + y + z = 0 \\ x + \lambda y + z = 0 \\ x + y + \lambda z = 0 \end{array}\right\} \to (1)$$

Then system (1) can be expressed in the matrix form $AX = 0$

$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Given that given system has a non trivial solution

$\rho(A) < n$; $n = 3$

$\rho(A) < 3$

Given matrix is a 3×3 matrix so that

$|A| = 0$

$$|A| = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2 - 1) - 1(\lambda - 1) + 1(1 - \lambda) = 0$$

$$\lambda^3 - \lambda - \lambda + 1 + 1 - \lambda = 0$$

$$\lambda^3 - 3\lambda + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$(\lambda - 1)(\lambda^2 + \lambda - 2) = 0$$

$$(\lambda - 1)(\lambda^2 + 2\lambda - \lambda - 2) = 0$$

$$(\lambda - 1)(\lambda(\lambda + 2) - 1(\lambda + 2)) = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 1, 1, -2$$

case (i)

$\lambda = 1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$\therefore \rho(A) = 1, \; n = 3$

$\rho(A) < n$

$n - r = 3 - 1 = 2 \quad$ L.I.S

$x + y + z = 0$

$y = k_1; \; z = k_2$

$x + k_1 + k_2 = 0$

$x = -(2k_1 + k_2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(2k_1 + k_2) \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ For $\lambda = 1$ the system has a non trivial solution

case (ii)

$\lambda = -2$

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$

$\rho(A) = 2; \; n = 3$

$\therefore$ For $\lambda = -2$ the system has a non trivial solution

$\rho(A) < n$

$n - r = 3 - 2 = 1 \quad$ L.I.S

$z = k$

$-2x + y + \cancel{z} = 0$

$-3\cancel{y} + 3\cancel{z} = 0$

$-3y + 3k = 0$

$-\cancel{3}y = -\cancel{3}k$

$y = k$

$$-2x+k+k=0$$
$$-\cancel{2}x=-\cancel{2}k$$
$$x=k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

H.W

6 Show that the only real number λ for which the system $x+2y+3z=\lambda x$; $3x+y+2z=\lambda y$; $2x+3y+z=\lambda z$ has non-zero solution is 6 and solve them when $\lambda=6$

Solu: Given system can be expressed as $AX=0$ where

$$A=\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} ; X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } 0=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here number of variables $n=3$

The given system of equations possess a non-zero solution, if

Rank of A $< n$

$$\rho(A)<3$$

For this $|A|=0$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}=0$$

Applying $R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$ 7.

$$(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -(\lambda+2) & -1 \\ 2 & 1 & -(\lambda+1) \end{vmatrix} \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix} = 0$$ 8.

$$(6-\lambda)\left[1((\lambda+2)(\lambda+1)+1) - 0(-3(\lambda+1)-2) + 0(3+2(\lambda+1)\right] = 0$$

$$(6-\lambda)\left[\lambda^2+2\lambda+\lambda+2+1-0+0\right] = 0 \Rightarrow (6-\lambda)[\lambda^2+3\lambda+3] =$$

$$(6-\lambda)[\lambda^2+3\lambda+3] =$$

$$\therefore \lambda = 6$$

7 Solu

$$(6-\lambda)\left[\cancel{4\lambda+3}\right] = 0$$

$$\left[24\lambda - 4\lambda^2 + 18 - 3\lambda = 0\right.$$

$$21\lambda + 18 = 4\lambda^2$$

$$\left.4\lambda^2 - 21\lambda - 18 = 0\right]$$

$\therefore$ Here $\lambda = 6$ is the only real value and other value are complex. when $\lambda = 6$, the given system becomes

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{matrix} R_2 \to 5R_2 + 3R_1 \\ R_3 \to 5R_3 + 2R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x + 2y + 3z = 0$; $-5x + 2k + 3k = 0$

$-19y + 19z = 0$; $\cancel{-}5x = \cancel{-}5k$

$z = k$; $x = k$

$-19y + 19k = 0$

$-19y = -19k$

$y = k$;

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

Gauss -

Solutions of Linear systems Direct Methods

1) Gaussian Elimination Method

This method of solving system of n linear Equations in 'n' unknowns consists of eliminating the co-efficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

1. Solve the Equations $2x+y+z=10; 3x+2y+3z=18; x+4y+9z=16$; by using Gauss elimination method.

Solu) Given Equations

$$\left.\begin{array}{l} 2x+y+z=10 \\ 3x+2y+3z=18 \\ x+4y+9z=16 \end{array}\right\} \rightarrow (1)$$

system (1) can be expressed in the form $AX=B$

where

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}; \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad B=\begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

Argumented matrix

$$[AB]=\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

$$\sim\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow 2R_2-3R_1 \\ R_3 \rightarrow 2R_3-R_1 \end{array}$$

$$\sim\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix} \quad R_3 \rightarrow R_3-7R_2$$

$$\begin{array}{r} 70 \\ 21 \\ \hline 59 \end{array} \qquad \begin{array}{r} 42 \\ 22 \\ \hline 20 \end{array}$$

which is a upper triangular matrix

$2x+y+z=10; \; y+3z=6$

$-4z=-20$

$z=5$

$y+3(5)=6 \quad ; \quad 2x-9+5=10$

$y=6-15 \qquad 2x=14$

$y=-9 \qquad x=7;$

$x=7; \; y=-9; \; z=5$

2. Solve $3x+y-z=3$; $2x-8y+z=-5$; $x-2y+9z=8$ by Gaussian elimination method

Given Equations

$$\left.\begin{array}{l} 3x+y-z=3 \\ 2x-8y+z=-5 \\ x-2y+9z=8 \end{array}\right\} \rightarrow ①$$

system ① can be expressed in the form $AX=B$

$$A=\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix}; \; X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \; B=\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

Argumented matrix

$$[AB]=\begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$$

$$\sim\begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & -7 & 28 & 21 \end{bmatrix} \begin{array}{l} R_2 \rightarrow 3R_2-2R_1 \\ R_3 \rightarrow 3R_3-R_1 \end{array}$$

-24
-2

$$\sim\begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & -1 & 4 & 3 \end{bmatrix} R_3 \rightarrow \frac{R_3}{7}$$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & 0 & 99 & 99 \end{bmatrix} \quad R_3 \rightarrow 26R_3 + R_2$$

$26 \times 3 = 78$, $78 + 21 = 99$; $26 \times 4 = 104$, $104 - 5 = 99$

which is a upper triangular matrix

$3x + y - z = 3$

$-26y + 5z = -21$

$\cancel{99}z = \cancel{99}$

$z = 1$

$3x + \cancel{1} - \cancel{1} = 3$

$x = 1$

$-26y + 5 = -21$

$-26y = -21 - 5$

$\cancel{-26}y = \cancel{-26}$

$y = 1$

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$\therefore x = 1, \; y = 1, \; z = 1$

3. Solve $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$ by using Gauss-Jordan-Method (only row operations)

Solu) Given Equations

$$\left.\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}\right\} \rightarrow (1)$$

system (1) can be expressed in the form $AX = B$ where

$$[A\ B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix} \quad \begin{aligned} R_2 &\rightarrow 2R_2 - 3R_1 \\ R_3 &\rightarrow 2R_3 - R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix} \quad R_3 \rightarrow R_3 - 7R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_3 \to R_3/-4$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 1 & 0 & -9 \\ 0 & 1 & 1 & 5 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_1 \to R_1/2$$

$x = 7$; $y = -9$; $z = 5$

H.W.

4. Solve the equations $x+y+z=6$; $3x+3y+4z=20$; $2x+y+3z=13$; using partial pivoting Gaussian elimination method.

Solu) Given Equations

$$\left.\begin{aligned} x+y+z &= 6 \\ 3x+3y+4z &= 20 \\ 2x+y+3z &= 13 \end{aligned}\right\} \to (1)$$

system (1) can be expressed in the form

$AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}.$$

Argumented matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_2 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

which is a upper triangular matrix

$x+y+z=6$; $x+1+2=6$, $x=3$

$-y+z=1$; $-y+2=1$, $-y=-1$, $y=1$;

$z=2$

$\therefore x=3$; $y=1$; $z=2$

5. Solve the Equations $3x+y+2z=3$; $2x-3y-z=-3$; $x+2y+z=4$ by using Gauss Elimination method

solu) Given Equations

$$\left.\begin{aligned} 3x+y+2z&=3 \\ 2x-3y-z&=-3 \\ x+2y+z&=4 \end{aligned}\right\} \to (1)$$

system (1) can be expressed in the form $AX=B$.

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

Argumented matrix

$$[AB] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} \quad R_3 \to 7R_3 - 5R_2.$$

which is an upper triangular matrix

$x + 2y + z = 4$; $x + 2(2) - 1 = 4$; $x + 4 - 1 = 4$; $x = 1$

$-7y - 3z = -11$; $-7y - 3(-1) = -11$; $-7y + 3 = -11$; $-7y = -14$; $y = 2$

$8z = -8$; $z = -1$

$\therefore x = 1 ; y = 2 ; z = -1$

6. Solve the equations $10x + y + z = 12$; $2x + 10y + z = 13$ and $x + y + 5z = 7$ by Gauss - Jordon Method

<u>Solu</u> Given Equations

$$\left.\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned}\right\} \to (1)$$

system (1) can be expressed in the form

$AX = B$

$$A = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; B = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

Argumented matrix

$$[AB] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & 49 & 4 & 53 \\ 0 & 9 & 49 & 58 \end{bmatrix} \quad \begin{aligned} R_2 &\to 5R_2 - R_1 \\ R_3 &\to 10R_3 - R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 1 \\ 0 & 49 & 4 & 53 \\ 0 & 0 & 2365 & 2365 \end{bmatrix} \quad R_3 \to 49R_3 - 9R_2$$

$$\sim \begin{bmatrix} 20 & 1 & 1 & 1 \end{bmatrix} \quad R_2 \leftrightarrow$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 10R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -8 & -44 & -51 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{bmatrix} \quad R_1 \rightarrow R_1 + R_3$$

$$\sim \begin{bmatrix} -1 & +8 & +44 & +51 \\ 0 & 8 & -9 & -1 \\ 0 & 9 & 49 & 58 \end{bmatrix} \quad \begin{matrix} R_1 \rightarrow \frac{R_1}{-1} \\ R_3 \rightarrow \frac{R_3}{-1} \end{matrix}$$

392
464

$$\sim \begin{bmatrix} -1 & 8 & 44 & 51 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & 473 & 473 \end{bmatrix} \quad R_3 \rightarrow 8R_3 - 9R_2$$

$$\sim \begin{bmatrix} -1 & 8 & 44 & 51 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_3 \rightarrow \frac{R_3}{473}$$

$$\sim \begin{bmatrix} -1 & 0 & 53 & 52 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 + 9R_3 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 53 & 52 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow \frac{R_2}{8}$$

$$\sim \begin{bmatrix} +1 & 0 & 0 & +1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \rightarrow \frac{R_1 - 53R_3}{-1}$$

$\therefore x = 1 ; \; y = 1 ; \; z = 1$

7. Solve the Equations
$10x_1 + x_2 + x_3 = 12$; $x_1 + 10x_2 - x_3 = 10$ and $x_1 - 2x_2 + 10x_3 = 9$ by Gauss - Jordan method

Date
15/12/18

2. Eigen Values Eigen Vectors & Quadratic

Let $A = [a_{ij}]_{m \times n}$ matrix a, non-zero vector x is said to be characteristics vector of A if there exsist a scalar λ such that $Ax = \lambda x$. If $Ax = \lambda X$, $(X \neq 0)$ we say that x is Eigen vector or characteristic vector of A corresponding to the Eigen values or characteristic vectors or values $\lambda(A)$

Note: $A - \lambda I$ is called characteristic matrix of A. also determinant $A - \lambda I$ is a polynomial in λ are degree 'n'

* $|A - \lambda I| = 0$ is called the characteristic equation of A. This will be polynomial equation in λ of degree 'n'. Here 'A' is $n \times n$ matrix (square matrix) & I is the $n \times n$ unit matrix i.e., should be satisfied

1. Find the Eigen values and Eigen vectors of the following matrix

i) $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Solu) Given matrix

$$A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix}$$

The characteristic equation of A is

$|A-\lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = 0$$

$(5-\lambda)[(6-\lambda)(7-\lambda)-4]+2[-2(7-\lambda)-0]+0=0$

$(5-\lambda)(42-13\lambda+\lambda^2-4)-28+4\lambda=0$

$(5-\lambda)(\lambda^2-13\lambda+38)-28+4\lambda=0$

$5\lambda^2-65\lambda+190-\lambda^3+13\lambda^2-38\lambda-28+4\lambda=0$

$-\lambda^3+18\lambda^2-99\lambda+162=0$

$\lambda^3-18\lambda^2+99\lambda-162=0$

$\lambda=3 \Rightarrow 27-162+297-162=0.$

$$\begin{array}{c|cccc} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & \boxed{0} \end{array}$$

$(\lambda^2-15\lambda+54)(\lambda-3)=0$

$\lambda-3=0$ | $\lambda^2-15\lambda+54=0$

$\lambda=3$ | $(\lambda-6)(\lambda-9)=0$

$\lambda=6,9$

$\therefore \lambda = 6, 9, 3$, are the characteristics of A or Eigen values or roots of A

<u>case(I)</u>

If $\lambda=3$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 / 2 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$2x - 2y = 0$; $\qquad \rho(A) = 2$; $n = 3$

$y + 2z = 0$ $\qquad$ let $z = k$

$n - r = 3 - 2 = 1$ L.I.S $\qquad y + 2k = 0$

$\qquad y = -2k$

$2x - 2(-2k) = 0$

$2x = -4k$

$x = -2k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

Case-II

If $\lambda = 6$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to \frac{R_2}{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 - R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$-x - 2y = 0$; $2y + z = 0$; $z = K$

$-x + 2\left(\frac{K}{2}\right) = 0$ $\quad$ $2y + K = 0$

$2y = -K$

$-x = -2K$ $\quad$ $y = \frac{-K}{2}$

$x = K$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K \\ -K/2 \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$$

<u>Case - III</u>

If $\lambda = 9$ then $(A - \lambda I) x = 0$

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ -2 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_1 \to \frac{R_1}{-2} \quad R_3 \to \frac{R_3}{2} \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_2 \to R_2 + R_1$$

R

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_2 \to \frac{R_2}{2} \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$2x + y = 0$; $z = K$

$-y + z = 0$ $\quad$ $2x + K = 0$

$-y + K = 0$ $\quad$ $2x = -K$

$-y = -K$ $\quad$ $x = \frac{-K}{2}$

$y = K$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ iii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

ii)

Solu| Given matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -(2+\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)[-(2-\lambda)(2+\lambda) - 0] - 2[0] - 1(0-0) = 0$$

$$(1-\lambda)[-(4 - \cancel{2\lambda} + \cancel{2\lambda} - \lambda^2)] = 0$$

$$(1-\lambda)[\cancel{2\lambda} - 4 + \lambda^2] = 0$$

$$(\lambda^2 - 4)(1-\lambda) = 0$$

$$\lambda^2 = 4 \quad 1 = \lambda$$

$$\lambda = \pm 2$$

$$\lambda = 1, 2, -2$$

$\therefore \lambda = 1, 2, -2$ are the Eigen roots of A

case (I)

If $\lambda = 1$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to 2R_2 - R_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to 5R_3 + 3R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$2y - z = 0$; $5z = 0$; $x = k$

$2y - 0 = 0$ $\quad z = 0$

$2y = 0$

$y = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

case (ii) If $\lambda = 2$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to 2R_3 + 4R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$-x+2y-z=0$

$2z=0 \quad ; x=k$

$z=0$

$-k+2y-0=0$

$2y=k$

$y=\frac{k}{2}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k/2 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

<u>Case-III</u>

If $\lambda=-2$ then $(A-\lambda I)X=0$

$$= \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to \frac{R_2}{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=2; n=3$ $\quad 3x+2y-z=0$ $\quad z=k \; ; \; 3x+2\left(\frac{-k}{2}\right)-k=0$

$n-r=3-2$ $\quad 2y+z=0$ $\quad 3x-2k=0$

$=1$ $\quad 2y+k=0$ $\quad 3x=2k$

L.I.S $\quad 2y=-k$ $\quad x=\frac{2}{3}k$

$y=-\frac{k}{2}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}k \\ -k/2 \\ k \end{bmatrix} = k\begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$$

3. Given matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristics matrix of A is

$$A-\lambda I = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2-\lambda & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$-(2+\lambda)[(-\lambda)(1-\lambda)-12] - 2(-2\lambda-6) - 3(-4+(1-\lambda)) = 0$$

$$-(2+\lambda)[-\lambda+\lambda^2-12] + 4\lambda + 12 - 3(-4+1-\lambda) = 0$$

$$-(2+\lambda)[-\lambda+\lambda^2-12] + 4\lambda + 12 - 3(-\lambda-3) = 0$$

$$-(2+\lambda)[\lambda^2-\lambda-12] + 4\lambda + 12 + 3\lambda + 9 = 0$$

$$-(2\lambda^2-2\lambda-24+\lambda^3-\lambda^2-12\lambda) + 7\lambda + 21 = 0$$

$$-2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+7\lambda+21 = 0$$

$$-\lambda^3-\lambda^2+21\lambda+45 = 0$$

$$\lambda^3+\lambda^2-21\lambda-45 = 0$$

$$\lambda = -3 \quad \begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$(\lambda+3)(\lambda^2-2\lambda-15) = 0$$

$$(\lambda+3)(\lambda^2-5\lambda+3\lambda-15) = 0$$

$$(\lambda+3)(\lambda(\lambda-5)+3(\lambda-5)) = 0$$

$$(\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$\therefore \lambda = -3, -3, 5$ are the Eigen roots of A

Case I

If $\lambda = -3$ $(A-\lambda I)X = 0$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 1$; $n = 3$

$n - r = 3 - 1 = 2$ L.I.S

$x + 2y - 3z = 0$

$y = k_1$; $z = k_2$

$x + 2k_1 - 3k_2 = 0$

$x = 3k_2 - 2k_1$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

case (II)

If $\lambda = 5$ then $(A-\lambda I)X = 0$

$$\sim \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} +1 & +2 & +5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_3 \to \frac{R_3}{-1} \\ R_1 \leftrightarrow R_3 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} +1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \to \frac{R_2}{-8} \\ R_3 \to \frac{R_3}{16} \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 - R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2 \; ; \; n = 3$

$n - r = 3 - 2 = 1$. L.I.S

$x + 2y + 5z = 0$; $x + 2(-2k) + 5(k) = 0$

$y + 2z = 0$; $x - 4k + 5k = 0$

$z = k$; $x = -k$

$y + 2k = 0$; $y = -2k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

Date 8/12/2018

Properties of Eigen Values:

1. The sum of the Eigen values of a square matrix is equal to its trace and product of the Eigen values is equals to its determinant
2. If 'λ' is an Eigen value of A corresponding to the Eigen vector "x" then λ^n is Eigen value of A^n corresponding to the Eigen vector "x"
3. A square matrix "A" and its transpose A^T have the same Eigen values.
4. If A and B are $n \times n$ matrix and if A is invertible then $A^{-1}B$ and BA^{-1} have same Eigen values.
5. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of matrix A
6. If $k\lambda_1 = k\lambda_2 \ldots, k\lambda_n$ are the Eigen values of matrix kA
7. If "λ" is the Eigen value of the matrix A then

$\lambda+K$ is an Eigen value of the matrix $A+KI$

8. If "λ" is an Eigen value of a non-singular matrix of A corresponding to the Eigen vector "x", then λ^{-1} is an Eigen value of A^{-1} and the corresponding Eigen values itself.

HW

2. Find the characteristic roots & characteristic vectors of the following matrices.

1. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Solu| 5. Given matrix

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -6-\lambda & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix}$$

The characteristic Equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -(3+\lambda) \end{vmatrix} = 0$$

$(1-\lambda)\,[(4-\lambda)(3+\lambda)+12]+6(0)-4(0)=0$

$(1-\lambda)\,[-(12-3\lambda+4\lambda-\lambda^2)+12]=0$

$(1-\lambda)\,[\lambda^2-\cancel{\lambda}-\cancel{12}+\cancel{12}]=0$

$(\lambda^2-\lambda)(1-\lambda)=0$

$\lambda^2-\lambda-\lambda^3+\lambda^2=0$

$\lambda^3-2\lambda^2+\lambda=0$

$$\begin{array}{c|cccc} 1 & 1 & -2 & 1 & 0 \\ & 0 & 1 & -1 & \\ \hline & 1 & -1 & \boxed{0} & \end{array}$$

$(\lambda^2-\lambda)(\lambda-1)=0$

$\lambda=1\;;\;\lambda^2=\lambda$

$\lambda=1$

$\therefore\ \lambda=1,1,0$ are the Eigen values of A

Case(i)

If $\lambda=1$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & -2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} R_3\to R_3-R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=2\;;\;n=3$

$n-r=3-2$. L.I.S

$-6y-4z=0$ $\quad z=k\;;\;-6(-\frac{2}{3}k)-41$

$3y+2z=0;$

$3y+2k=0$

$3y=-2k$

$y=-\frac{2}{3}k$

$$\begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2\to 2R_2+R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 1$; $n = 3$

$n - r = 3 - 1 = 2$, L.I.S

$-6y - 4z = 0$; $x = k_1$, $z = k_2$

$-6y - 4k_2 = 0$

$-6y = 4k_2$

$y = -\frac{2}{3}k_2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ -\frac{2}{3}k_2 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

<u>case (ii)</u>

If $\lambda = 0$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to 4R_3 + 6R_2$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$. L.I.S

$x - 6y - 4z = 0$; $z = k$

$4y + 2z = 0$

$4y + 2k = 0$; $x - 6\left(-\frac{1}{2}k\right) - 4k = 0$

$4y = -2k$; $x + 3k - 4k = 0$

$y = -\frac{1}{2}k$; $x = k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -1/2\,k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$$

1. Given matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -9 & 3 \end{bmatrix}$$

the characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

the characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right] + 1(-3+\lambda+1) + 1(1-5+\lambda) = 0$$

$$(3-\lambda)\left[15-3\lambda-5\lambda+\lambda^2-1\right] - 3+\lambda+1+1-5+\lambda = 0$$

$$45-9\lambda-15\lambda+3\lambda^2-3-15\lambda+3\lambda^2+5\lambda^2-\lambda^3+\lambda+2$$
$$-3+\lambda-5+\lambda = 0$$

$$-\lambda^3+6\lambda^2+5\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$\lambda = 3 \quad \begin{array}{c|cccc} 3 & 1 & -11 & 36 & -36 \\ & 0 & 3 & -24 & 36 \\ \hline & 1 & -8 & +12 & 0 \end{array}$$

$$(\lambda-3)(\lambda^2-8\lambda+12) = 0$$

$$(\lambda-3)(\lambda-2)(\lambda-6) = 0$$

$$\lambda = 2, 3, 6$$

Case (i)

If $\lambda = 2$ then $(A-\lambda I)x = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$; L.I.S

$x - y + z = 0$; $z = k$

$2y = 0$ $x - 0 + k = 0$

$y = 0$ $x = -k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii)

If $\lambda = 3$ then $(A-\lambda I)x = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 + R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to R_2 - R_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y+z=0$ $\therefore \rho(A)=2;\ n=3$

$n-r=3-2=1$ L.I.S

$-x+y=0$; $z=k$

$-y+k=0$; $-x+k=0$

$-y=-k$; $-x=-k$

$y=k$; $x=k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case (iii)

If $\lambda = 6$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -2 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to 2R_3 - 4R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=3$; $n=3$

$-3x-y+z=0$; $-2y-4(0)=0$; $-3x-0+0=0$

$-2y-4z=0$; $-2y=0$; $-3x=0$

$14z=0$; $y=0$; $x=0$

$z=0$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1\. Given matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)[(3-\lambda)(3-\lambda)-1] + 2[(-2)(3-\lambda)+2] + 2[2-6+2\lambda] = 0$$

$$(6-\lambda)[9-3\lambda-3\lambda+\lambda^2-1] + 2[-6+2\lambda+2] + 2[-4+2\lambda] = 0$$

$$54-18\lambda-18\lambda+6\lambda^2-6-9\lambda+3\lambda^2+3\lambda^2-\lambda^3+\lambda-12+4\lambda+4$$
$$-8+4\lambda = 0$$

$$-\lambda^3+12\lambda^2-36\lambda+32=0$$

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$\lambda = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \underline{0} \end{array}$$

$$(\lambda^2-10\lambda+16)(\lambda-2)=0$$

$$(\lambda^2-2\lambda-8\lambda+16)(\lambda-2)=0$$

$$(\lambda-2)[(\lambda-2)\lambda-8(\lambda-2)]=0$$

$$(\lambda-2)(\lambda-8)(\lambda-2)=0$$

$$\lambda = 2, 2, 8$$

$\therefore \lambda = 2, 2, 8$ are the Eigen values

Case (i)

If $\lambda = 2$ then $(A-\lambda I)x=0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 1$; $n = 3$

$n - r = 3 - 1 = 2$; L.I.S

$4x - 2y + 2z = 0$

$y = k_1$; $z = k_2$

$4x - 2k_1 + 2k_2 = 0$

$4x = 2k_1 - 2k_2$

$x = \frac{k_1}{2} - \frac{k_2}{2}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1/2 - k_2/2 \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

case (ii)

If $\lambda = 8$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_1 \to \frac{R_1}{2} \\ R_3 \to R_3 - R_2 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x-y+z=0$

$-3y-3z=0 \quad ; z=k \quad ; \quad -x-(-k)+k=0$

$-3y-3k=0 \qquad\qquad -x+2k=0$

$-3y=3k \qquad\qquad -x=-2k$

$y=-k \qquad\qquad x=k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

2. Given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The choracteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

The choracteristic equation of A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$(8-\lambda)[(7-\lambda)(3-\lambda)-16]+6(-6(3-\lambda)+8)+2(24-2(7-\lambda))=0$

$(8-\lambda)[21-3\lambda-7\lambda+\lambda^2-16]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$

$(8-\lambda)[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda+10]=0$

$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$

$-\lambda^3+18\lambda^2-45\lambda=0$

$$\lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0 \;;\; (\lambda^2 - 15\lambda - 3\lambda + 45) = 0$$

$$[\lambda(\lambda - 15) - 3(\lambda - 15)]\lambda = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15$$

$\therefore \lambda = 0, 3, 15$ are the Eigen values

case (i)

If $\lambda = 0$ then $(A - \lambda I)X = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20 & -20 \\ 0 & -20 & 20 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{matrix} R_2 \to 8R_2 + 6R_1 \\ R_3 \to 8R_3 - 2R_1 \end{matrix} \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20 & -20 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 2 \;;\; n = 3$$

$$n - r = 3 - 2 = 1 \;;\; \text{L.I.S}$$

$$8x - 6y + 2z = 0 \qquad ; \; z = k$$

$$20y - 20z = 0$$

$$20y = 20z$$

$$y = z \;;\; y = k$$

$$8x - 6k + 2k = 0$$

$$8x - 4k = 0 \;;\; 8x = 4k \;;\; x = \frac{k}{2}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k/2 \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

case(ii)

If $\lambda = 3$ then $(A-\lambda I)X = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{matrix} R_2 \to 5R_2 + 6R_1 \\ R_3 \to 5R_3 - 2R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{R_2}{-8} \\ R_3 \to \frac{R_3}{-4} \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_3 \to R_3 - R_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$; L.I.S

$5x - 6y + 2z = 0$; $z = k$

$2y + z = 0$; $2y + k = 0$

$2y = -k$

$y = \frac{-k}{2}$

$5x - 6\left(\frac{-k}{2}\right) + 2k = 0$

$5x + 3k + 2k = 0$

$5x = -5k$

$x = -k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -k/2 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

case(iii)

If $\lambda = 15$ then $(A-\lambda I)X = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 3 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_2 \to \frac{R_2}{-2}, \; R_3 \to \frac{R_3}{-2} \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & 10 & 20 \\ 0 & 20 & 30 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_2 \to 7R_2 + 3R_1, \; R_3 \to 7R_3 - R_1 \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_2 \to R_2/10, \; R_3 \to R_3/10 \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad R_3 \to R_3 - 2R_2 \quad = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 3$; $n = 3$

$-7x - 6y + 2z = 0$; $y + 2z = 0$; $-z = 0$

$z = 0$

$x = 0$ $\qquad y = 0$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3. Given matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}=0$$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right]-1\left[\cancel{1}-\lambda-\cancel{1}\right]+1\left[\cancel{1}-\cancel{1}+\lambda\right]=0$$

$$(1-\lambda)\left[\cancel{1}-\lambda-\lambda+\lambda^2-\cancel{1}\right]-1\left[-\lambda\right]+\lambda=0$$

$$(1-\lambda)\left[-2\lambda+\lambda^2\right]+\lambda+\lambda=0$$

$$\cancel{-2\lambda}+\lambda^2+2\lambda^2-\lambda^3\cancel{+\lambda+\lambda}=0$$

$$-\lambda^3+3\lambda^2=0$$

$$-\lambda(\lambda^2-3\lambda)=0$$

$$\lambda=0 \quad \lambda^2=+3\lambda$$

$$\lambda=+3$$

$$\lambda=0,0,3$$

Case (i)

If $\lambda=0$ then $(A-\lambda I)X=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{array}{l} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=1$; $n=3$

$n-r=3-1=2$; L.I.S

$x+y+z=0$; $y=k_1$; $z=k_2$

$x+k_1+k_2=0$

$x=-(k_1+k_2)$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -(k_1+k_2) \\ k_1 \\ k_2 \end{bmatrix}=k_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+k_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$